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A SUBCLASS OF ANALYTIC FUNCTIONS WITH TWO FIXED POINTS

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ABSTRACT. Making use of operator of fractional calculus a subclass $\varphi(\alpha,\beta,\mu,\eta,\gamma,\delta,t;z_0)$ of univalent functions with fixed point in the unit disk E is introduced and obtained coefficient-estimates distortion theorem. Lastly we investigated Hadamard product property and linear combination function of $\varphi(\alpha,\beta,\mu,\eta,\gamma,\delta,t;z_0)$.

1. INTRODUCTION

Let A denote the class of functions of the form

(1.1) $$f(z)=z-\sum_{n=2}^{\infty}a_nz^n(a_n\geq 0),$$

which are analytic in unit disc $E=\{z:|z|<1\}$. Silvermann ([4]) studied the class of functions of the form

$$f(z)=a_1z-\sum_{n=2}^{\infty}a_nz^n\ (a_n\geq 0),$$

where either

(1.2) $$f(z_0)=z_0(-1<z_0<1;z_0\neq 0)\quad\text{or}\quad f'(z_0)=1(-1<z_0<1).$$

Recently, Uralegadi and Somanatha([6]) studied the class of functions of the form

(1.3) $$f(z)=a_1z-\sum_{n=2}^{\infty}a_nz^n\ (a_n\geq 0)\quad\text{with}\quad \frac{(1-t)\,f(z_0)}{z_0}+tf'(z_0)=1,$$

where $-1<z_0<1$, $0\leq t\leq 1$. A function $f(z)$ is said to be convex of order α, if

$$\Re\left\{1+\frac{zf''(z)}{f(z)}\right\}>\alpha\,(z\in E:0\leq\alpha<1).$$

We denote by $C^*(\alpha)$ the class of convex functions of order α $(0\leq\alpha<1)$.

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We now recall the fallowing definition of a generalized fractional operator introduced by Srivastava et al([5]).

Definition 1 For real numbers $\eta(\eta > 0), \gamma$ and δ, the generalized fractional integral operator $I_{0,z}^{\eta,\gamma,\delta}$ of order η is defined for a function $f(z)$, by

$$I_{0,z}^{\eta,\gamma,\delta} f(z) = \frac{z^{-\eta-1}}{\Gamma(\eta)} \int_0^z (z-\xi)^{\eta-1} F(\eta+\gamma, -\delta; \eta; 1-\frac{\xi}{z}) f(\xi) d\xi,$$

where $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin with the order

$$f(z) = O\left(|z|^{\epsilon}\right), (z \to 0), (\epsilon < \max\{0, \gamma, -\delta\} - 1),$$

$$F(a,b,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \quad (z \in D), \tag{1.4}$$

and $(v)_n$ being the pochhammer symbol defined by

$$(v)_n = \frac{\Gamma(v+n)}{\Gamma(v)},$$

if $n = 0$ then $(v)_n = 1$ and if $n \in N = \{1, 2, \cdots\}$ then $v_n = v(v+1)\cdots(v+n-1)$, provided further that the multiplicity of $(z-\xi)^{\eta-1}$ is removed requiring $\log(z-\xi)$ to be real $(z-\xi) > 0$.

Definition 2 For real numbers $\eta(0 \leq \eta < 1)$, γ, and δ, the generalized fractional derivative operator $J_{0,z}^{\eta,\gamma,\delta}$ of order η is defined for a function $f(z)$, by

(1.5)

$$J_{0,z}^{\eta,\gamma,\delta} f(z) = \frac{1}{\Gamma(1-\eta)} \frac{d}{dz} \left\{ z^{\eta-\gamma} \int_0^z (z-\eta)^{-\eta} F\left(\gamma-\eta, -\delta; 1-\eta; 1-\frac{\xi}{z}\right) f(\xi) d\xi \right\},$$

where $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\xi)^{-\eta}$ is removed as Definition 1 obove.

It follows readily form Definition 2, $J_{0,z}^{\eta,\eta,\delta} f(z) = D_z^{\eta} f(z)\, (0 \leq \eta < 1)$, where oprator D_z^{η} is fractional derivatives operator which is defined by Owa([2]). Furthermore, in terms of Gamma functions, we have the following Lemma.

Lemma 3. *([5]) If $0 \leq \eta < 1$ and $n > \gamma - \delta - 2$, then*

$$J_{0,z}^{\eta,\gamma,\delta} z^n = \frac{\Gamma(n+1)\Gamma(n-\gamma+\delta+2)}{\Gamma(n-\gamma+1)\Gamma(n-\eta+\delta+2)} z^{n-\gamma}. \tag{1.6}$$

Lemma 4. *If the form of a function $f(z)$ defined by (1.2) and satisfying (1.3), then*
(1.7)
$$\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} z^{\gamma-1} J_{0,z}^{\eta,\gamma,\delta} f(z) = a_1 - \sum_{n=2}^{\infty} \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n a_n z^{n-1},$$

where we denote $\Psi_n(\eta,\gamma,\delta) = \frac{\Gamma(n+1)\Gamma(n-\gamma+\delta+2)}{\Gamma(n-\gamma+1)\Gamma(n-\eta+\delta+2)}$.

proof. By Lemma3, we get a (1.7). □

We will define the following definition.

Definition 5 A function $f(z)$ defined by (1.2) and satisfying (1.3) is said to be in the class $\varphi(\alpha,\beta,\mu,\eta,\gamma,\delta,t;z_0)$ if

(1.8)
$$\left| \frac{\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} z^{\gamma-1} J_{0,z}^{\eta,\gamma,\delta} f(z) - a_1}{\mu\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} z^{\gamma-1} J_{0,z}^{\eta,\gamma,\delta} f(z) + a_1 - (1+\mu)\alpha} \right| < \beta, z \in E,$$

where

$$0 \le \eta < 1, \eta-\delta < 3, \gamma-\delta < 3, 0 \le \mu \le 1, 0 \le \alpha < 1, 0 < \beta \le 1 \text{ and } \alpha < a_1.$$

Furthermore, by specializing the parameters $\alpha,\beta,\mu,\eta,\gamma,\delta,t$, we obtain the following subclasses studied by various authors,

(1) $\varphi(\alpha,\beta,\mu,\eta,\eta,\delta,1;0) = P^*(\alpha,\beta,\mu,\eta)$(Jochi [1]);
(2) $\varphi(\alpha,\beta,\mu,1,1,\delta,1;0) = P^*(\alpha,\beta,\mu)$ (Owa and Aouf [3]);

The main purpose of this paper is to investigate coefficient-inequalites, distortion theorem and radius problem of functions in the class $\varphi(\alpha,\beta,\mu,\eta,\gamma,\delta,t;z_0)$. And, we obtain Hadmard product property and linear combination function.

2. A Coefficient Theorem

We begin by starting our first result as,

Theorem 2.1. *A function $f(z)$ is in the class $\varphi(\alpha,\beta,\mu,\eta,\gamma,t,\delta;z_0)$ if and only if*
(2.1)
$$\sum_{n=2}^{\infty} \left\{ (1+\mu\beta) \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n - [(1-t)+tn] z_0^{n-1} \right\} a_n \le (1+\mu)\beta(1-\alpha),$$
where $\Psi_n(\mu,\gamma,\delta)$ is in (1.7).

proof. Suppose that $f(z)$ is in the class $\varphi(\alpha,\beta,\mu,\eta,\gamma,t,\delta;z_0)$, so that condition (1.8) readily yields.

$$\left| \frac{\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} z^{\gamma-1} J_{0,z}^{\eta,\gamma,\delta} f(z) - a_1}{\mu\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} z^{\gamma-1} J_{0,z}^{\eta,\gamma,\delta} f(z) + a_1 - (1+\mu)\alpha} \right| < \beta.$$

Using Lemma 4, we obtain that

$$\left|\frac{-\sum_{n=2}^{\infty}\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)}\Psi_n a_n z^{n-1}}{\mu\left\{a_1-\sum_{n=2}^{\infty}\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)}\Psi_n a_n z^{n-1}\right\}+a_1-(1+\mu)\alpha}\right|<\beta\ (z\in E).$$

Since $|\Re(z)|\leq|z|$, for any z, we have

$$\Re\left\{\frac{\sum_{n=2}^{\infty}\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)}\Psi_n a_n z^{n-1}}{(1+\mu)(a_1-\alpha)-\mu\sum_{n=2}^{\infty}\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)}\Psi_n a_n z^{n-1}}\right\}<\beta. \tag{2.2}$$

Choose values of z on the real axis so that $\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)}z^{\gamma-1}J_{0,z}^{\eta,\gamma,\delta}f(z)$ is real, upon clearing the denominator in (2.2) and letting $z\to 1$ through the real values, we get

$$\sum_{n=2}^{\infty}(1+\mu\beta)\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)}\Psi_n a_n\leq(1+\mu)\beta(a_1-\alpha). \tag{2.3}$$

Finally, substituting $a_1=1+\sum_{n=2}^{\infty}[(1-t)+tn]\,a_n|z_0|^{n-1}$ in (2.3), we get (2.1).

Conversely, assume that the inequality (2.1) holds true. Consider

$$\begin{aligned}
&\left|\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)}z^{\gamma-1}J_{0,z}^{\eta,\gamma,\delta}f(z)-a_1\right|-\\
&\beta\left|\mu\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)}z^{\gamma-1}J_{0,z}^{\eta,\gamma,\delta}f(z)+a_1-(1+\mu)\alpha\right|\\
&\leq\sum_{n=2}^{\infty}\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)}\Psi_n a_n|z|^{n-1}\\
&-(1+\mu)\beta(a_1-\alpha)+\beta\mu\sum_{n=2}^{\infty}\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)}\Psi_n a_n|z|^{n-1}\\
&\leq\sum_{n=2}^{\infty}(1+\mu\beta)\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)}\Psi_n a_n-(1+\mu)\beta(a_1-\alpha)\leq 0,
\end{aligned}$$

by the hypothesis. Hence, a function $f(z)$ is in the class $\varphi(\alpha,\beta,\mu,\eta,\gamma,t,\delta;z_0)$. □

Corollary 2.2. *Let the function $f(z)=a_1z-\sum_{n=2}^{\infty}a_nz^n$ defined by (1.3) be in the class $\varphi(\alpha,\beta,\mu,\eta,\gamma,\delta,t;z_0)$. Then*

$$a_n\leq\frac{(1+\mu)\beta(1-\alpha)}{(1+\mu\beta)\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)}\Psi_n+(1-a_1)}. \tag{2.4}$$

The assertion (2.1) of Theorem2.1 is sharp extremal function being

$$f(z) = a_1 z - \frac{(1+\mu)\,\beta\,(1-\alpha)}{(1+\mu\beta)\,\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)}\Psi_n + 1 - a_1} z^n. \tag{2.5}$$

Where $a_1 = 1 + \sum\limits_{n=2}^{\infty} \left[(1-t) + tn\right] a_n z_0^{n-1}$.

3. Distortion Theorem

Theorem 3.1. *If a function* $f(z)$ *is in the class* $\varphi(\alpha, \beta, \mu, \eta, \gamma, \delta, t; z_0)$ *with* $\eta > \gamma$, *then*

$$|f(z)| \geq a_1 |z| - \frac{(2-\gamma)(3-\eta+\delta)(1+\mu)\,\beta\,(a_1-\alpha)}{2(3-\gamma+\delta)(1+\mu\beta)} |z|^2, \tag{3.1}$$

and

$$|f(z)| \leq a_1 |z| + \frac{(2-\gamma)(3-\eta+\delta)(1+\mu)\,\beta\,(a_1-\alpha)}{2(3-\gamma+\delta)(1+\mu\beta)} |z|^2. \tag{3.2}$$

proof. In view of inequality (2.1) and the fact that $\Psi_n(\eta, \gamma, \delta)$ is non-decreasing for $\eta \geq \gamma$, we have

$$\begin{aligned}
&(1+\mu)\,\beta\,(a_1-\alpha) \\
&\geq \sum_{n=2}^{\infty} (1+\mu\beta) \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \frac{\Gamma(n+1)\Gamma(n-\gamma+\delta+2)}{\Gamma(n-\gamma+1)\Gamma(n-\eta+\delta+2)} a_n \\
&\geq \frac{2(3-\gamma+\delta)(1+\mu\beta)}{(2-\gamma)(3-\eta+\delta)} \sum_{n=2}^{\infty} a_n.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
|f(z)| &\geq a_1 |z| - \sum_{n=p+1}^{\infty} a_n |z|^n \geq a_1 |z| - |z|^2 \sum_{n=p+1}^{\infty} a_n \\
&\geq a_1 |z| - \frac{(2-\gamma)(3-\eta+\delta)(1+\mu)\,\beta\,(a_1-\alpha)}{2(3-\gamma+\delta)(1+\mu\beta)} |z|^2,
\end{aligned}$$

$$\sum_{n=2}^{\infty} a_n \leq \frac{(2-\gamma)(3-\eta+\delta)(1+\mu)\,\beta\,(a_1-\alpha)}{2(3-\gamma+\delta)(1+\mu\beta)}, \tag{3.3}$$

and we have

$$|f(z)| \geq a_1 |z| - \frac{(2-\gamma)(3-\eta+\delta)(1+\mu)\,\beta\,(a_1-\alpha)}{2(3-\gamma+\delta)(1+\mu\beta)} |z|^2.$$

Simillary,

$$|f(z)| \leq a_1 |z| + \frac{(2-\gamma)(3-\eta+\delta)(1+\mu)\,\beta\,(a_1-\alpha)}{2(3-\gamma+\delta)(1+\mu\beta)} |z|^2.$$

The proof is complete. □

Theorem 3.2. *If a function $f(z)$ is in the class $\varphi(\alpha,\beta,\mu,\eta,\gamma,\delta,t;z_0)$, then*

$$(3.4)\quad \left|J_{0,z}^{\eta,\gamma,\delta}f(z)\right| \geq \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)}\left\{a_1|z|^{1-\gamma}-\frac{(1+\mu)\beta(a_1-\alpha)}{(1+\mu\beta)}|z|^{2-\gamma}\right\},$$

and

$$(3.5)\quad \left|J_{0,z}^{\eta,\gamma,\delta}f(z)\right| \leq \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)}\left\{a_1|z|^{1-\gamma}+\frac{(1+\mu)\beta(a_1-\alpha)}{(1+\mu\beta)}|z|^{2-\gamma}\right\},$$

for $z\in D_0$ where D_0 equals to E if $\gamma\leq 1$, and D_0 equals to E^* if $1<\gamma<n$.

proof. By using in equality (1.8) and Theorem 2.1, we obtain that

$$\left|\frac{\Gamma(3-\gamma+\delta)}{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}z^{\gamma}J_{0,z}^{\eta,\gamma,\delta}f(z)\right| \geq a_1|z|-\sum_{n=2}^{\infty}\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)}\Psi_n a_n|z|^n$$
$$\geq a_1|z|-|z|^2\sum_{n=2}^{\infty}\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)}\Psi_n a_n \geq a_1|z|-|z|^2\frac{(1+\mu)\beta(a_1-\alpha)}{(1+\mu\beta)},$$

which is equivalent to (3.4). Simillary, We obtain that

$$\left|J_{0,z}^{\eta,\gamma,\delta}f(z)\right| \leq \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)}\left\{a_1|z|^{1-\gamma}+\frac{(1+\mu)\beta(a_1-\alpha)}{(1+\mu\beta)}|z|^{2-\gamma}\right\}.$$

The proof is complete. □

Corollary 3.3. *Let a function $f(z)$ be in the class $\varphi(\alpha,\beta,\mu,\eta,\gamma,\delta,t;z_0)$ with $\eta>\gamma$. Then, in view of Theorem 3.1, $f(z)$ is included in a disk with its center at origin and radius r given by*

$$(3.6)\qquad r=a_1+\frac{(2-\gamma)(3-\eta+\delta)}{2(3-\gamma+\delta)}\frac{(1+\mu)\beta(a_1-\alpha)}{(1+\mu\beta)},$$

and $J_{0,z}^{\eta,\gamma,\delta}f(z)$ is included in a disk with its center at origin and radius R given by

$$(3.7)\qquad R=\frac{\Gamma(3-\gamma+\delta)}{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}\left\{a_1+\frac{(1+\mu)\beta(a_1-\alpha)}{(1+\mu\beta)}\right\}.$$

4. Radius of convexity

Theorem 4.1. *Let $f(z)$ be in the class $\varphi(\alpha,\beta,\mu,\eta,\gamma,\delta,t;z_0)$. Then $f(z)$ is convex in the disk*
(4.1)
$$|z| < r(\alpha,\beta,\mu,\eta,\gamma,\delta,t;z_0)$$
$$= \inf_{\substack{n\geq 2\\ n\in N}} \left\{ \frac{(1+\mu\beta)\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)}\Psi_n + [(1+\mu)\beta(1-\alpha)-1][(1-t)+tn]z_0^{n-1}}{n^2(1+\mu)\beta(1-\alpha)} \right\}^{\frac{1}{n-1}}$$

The result is sharp for the function given by

proof. It is sufficient to prove that $\left|\frac{zf''(z)}{f'(z)}\right| \leq 1$ for $r(\alpha,\beta,\mu,\eta,\gamma,\delta,t;z_0)$.

A simple calculation gives us

$$\left|\frac{zf''(z)}{f'(z)}\right| = \left|\frac{-\sum\limits_{n=2}^{\infty} n(n-1)a_n z^{n-1}}{a_1 - \sum\limits_{n=2}^{\infty} n a_n z^{n-1}}\right| \leq \frac{\sum\limits_{n=2}^{\infty} n(n-1)a_n |z|^{n-1}}{a_1 - \sum\limits_{n=2}^{\infty} n a_n |z|^{n-1}}.$$

Clearly $\left|\frac{zf''(z)}{f'(z)}\right| \leq 1$, if

$$\sum_{n=2}^{\infty} n(n-1)a_n|z|^{n-1} \leq a_1 - \sum_{n=2}^{\infty} na_n|z|^{n-1}. \tag{4.3}$$

Using $a_1 = 1 + \sum\limits_{n=2}^{\infty}[(1-t)tn]a_1 z_0^{n-1}$ in (4.3), we are led to

$$\sum_{n=2}^{\infty}\left\{n^2|z|^{n-1} - [(1-t)+tn]z_0^{n-1}\right\}a_n \leq 1. \tag{4.4}$$

By Theorem 2.1, we have

$$\sum_{n=2}^{\infty}\left\{\frac{(1+\mu\beta)\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)}\Psi_n - [(1-t)+tn]z_0^{n-1}}{(1+\mu)\beta(1-\alpha)}\right\}a_n \leq 1. \tag{4.5}$$

Hence (4.4) will hold, if
(4.6)
$$n^2|z|^{n-1} - [(1-t)+tn]z_0^{n-1} \leq \frac{(1+\mu\beta)\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)}\Psi_n - [(1-t)+tn]z_0^{n-1}}{(1+\mu)\beta(1-\alpha)},$$

or equivalently

$$|z|^{n-1} \leq \frac{(1+\mu\beta)\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)}\Psi_n + [(1+\mu)\beta(1-\alpha)-1][(1-t)+tn]z_0^{n-1}}{n^2(1+\mu)\beta(1-\alpha)} \tag{4.7}$$

which in turm implies the assertion of the theorem. □

5. Property of Hadamard product

Let the function $f_j(z)\,(j=1,2)$ defined by $f_j(z) = a_{1,j}z - \sum_{n=2}^{\infty} a_{n,j}z^n$, we define the hadamard product $f(z) * g(z)$ by

$$(f_1 * f_2)(z) = a_{1,1}a_{1,2}z - \sum_{n=2}^{\infty} a_{n,1}a_{n,2}z^n. \tag{5.1}$$

Theorem 5.1. *Let the functions $f_j(z)(j = 1,2)$ defined by (5.1) be in the class $\varphi(\alpha,\beta,\mu,\eta,\gamma,\delta,t;z_0)$ with $\mu > \gamma$. Then $f_1(z)*f_2(z)$ is in the class $\varphi(v,\beta,\mu,\eta,\gamma,\delta,t;z_0)$ where*

$$v = v(\alpha,\beta,\mu,\eta,\gamma,\delta) = a_{1,1}a_{1,2} - \frac{(2-\gamma)(3-\eta+\delta)(1+\mu)\beta(a_{1,1}-\alpha)(a_{1,2}-\alpha)}{2(3-\gamma+\delta)(1+\mu\beta)}. \tag{5.2}$$

proof. Suppose that $f_1(z)$ and $f_2(z)$ are in the class $\varphi(\alpha,\beta,\mu,\eta,\gamma,\delta,t;z_0)$, by using Theorem 2.1, we have

$$\sum_{n=2}^{\infty} \frac{(1+\mu\beta)\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)}\Psi_n a_{n,1}}{(1+\mu)\beta(a_{1,1}-\alpha)} \le 1, \tag{5.3}$$

and

$$\sum_{n=2}^{\infty} \frac{(1+\mu\beta)\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)}\Psi_n a_{n,2}}{(1+\mu)\beta(a_{1,2}-\alpha)} \le 1. \tag{5.4}$$

From (5.3) and (5.4), using Cauchy-Schwarze inequality, we have

$$\sum_{n=2}^{\infty} \frac{(1+\mu\beta)\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)}\Psi_n}{(1+\mu)\beta} \sqrt{\frac{a_{n,1}a_{n,2}}{(a_{1,1}-\alpha)(a_{1,2}-\alpha)}} \le 1. \tag{5.5}$$

Hence, we find the largest v such that

$$\begin{aligned} &\sum_{n=2}^{\infty} \frac{(1+\mu\beta)\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)}\Psi_n}{(1+\mu)\beta(a_{1,1}a_{1,2}-v)} a_{n,1}a_{n,2} \\ &\le \sum_{n=2}^{\infty} \frac{(1+\mu\beta)\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)}\Psi_n}{(1+\mu)\beta} \sqrt{\frac{a_{n,1}a_{n,2}}{(a_{1,1}-\alpha)(a_{1,2}-\alpha)}} \le 1, \end{aligned} \tag{5.6}$$

or equivalently

$$\sqrt{a_{n,1}a_{n,2}} \leq \frac{a_{1,1}a_{1,2} - v}{\sqrt{(a_{1,1}-\alpha)(a_{1,2}-\alpha)}}. \tag{5.7}$$

So, it is sufficient to find the largest v such that

$$\frac{(1+\mu)\beta\sqrt{(a_{1,1}-\alpha)(a_{1,2}-\alpha)}}{(1+\mu\beta)\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)}\Psi_n} \leq \frac{a_{1,1}a_{1,2} - v}{\sqrt{(a_{1,1}-\alpha)(a_{1,2}-\alpha)}}. \tag{5.8}$$

Hence (5.8) yields $v \leq a_{1,1}a_{1,2} - \frac{(1+\mu)\beta(a_{1,1}-\alpha)(a_{1,2}-\alpha)}{(1+\mu\beta)\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)}\Psi_n}$. Since $\Psi_n(\mu,\gamma,\delta)$ is non-decreasing for $\mu \geq \gamma$, we have

$$v \leq a_{1,1}a_{1,2} - \frac{(2-\gamma)(3-\eta+\delta)(1+\mu)\beta(a_{1,1}-\alpha)(a_{1,2}-\alpha)}{2(3-\gamma+\delta)(1+\mu\beta)},$$

which proves the assertion of this theorem. □

6. Linear combination of the function in the class $\varphi(\alpha,\beta,\mu,\eta,\gamma,\delta,t;z_0)$

Theorem 6.1. *Let* $T(\alpha,\beta,\mu,\eta,\gamma,\delta) = \frac{(1+\mu)\beta(1-\alpha)}{(1+\mu\beta)\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)}\Psi_n + 1 - a_1}$, *and let us put*

$$f_n(z) = a_1 z - T(\alpha,\beta,\mu,\eta,\gamma,\delta)\, z^n, n = 2,3,\cdots, \tag{6.1}$$

and $f_1(z) = a_1 z$. *Then* $f(z) \in \varphi(\alpha_j,\beta_j,\mu_j,\eta,\gamma,\delta,t_j;z_0)$ *if and only if*

$$f(z) = \sum_{n=1}^{\infty} t_n f_n(z), z \in E, \tag{6.2}$$

where $\sum_{n=1}^{\infty} t_n = 1, t_n \geq 0$ *for* $n = 1,2,3,\cdots$.

proof. Let $f(z) \in \varphi(\alpha,\beta,\mu,\eta,\gamma,\delta,t;z_0)$. Then, by Corollary 2.2, $|a_n| \leq T(\alpha,\beta,\mu,\eta,\gamma,\delta)$. Let us put

$$t_n = T(\alpha,\beta,\mu,\eta,\gamma,\delta)^{-1} a_n, n = 2,3,\cdots, \tag{6.3}$$

and $t_1 = 1 - \sum_{n=2}^{\infty} t_n$. By assumption, we have $t_n \geq 0, n = 2,3,\cdots$, and $t_1 \geq 0$. Thus

$$\begin{aligned} \sum_{n=1}^{\infty} t_n f_n(z) &= t_1 f_1(z) + \sum_{n=2}^{\infty} t_n f_n(z) \\ &= \left(1 - \sum_{n=2}^{\infty} t_n\right) a_1 z + \sum_{n=2}^{\infty} t_n \left\{a_1 z - T(\alpha,\beta,\mu,\eta,\gamma,\delta)\, z^n\right\} \\ &= a_1 z - \sum_{n=2}^{\infty} a_n z^n = f(z). \end{aligned} \tag{6.4}$$

Conversely, Let us function $f(z)$ satisfy (6.2). Since

$$
\begin{aligned}
f(z) &= \sum_{n=1}^{\infty} t_n f_n(z) = t_1 f_1(z) + \sum_{n=2}^{\infty} t_n f_n(z) \\
&= \left(1 - \sum_{n=2}^{\infty} t_n\right) a_1 z + \sum_{n=2}^{\infty} t_n \left\{a_1 z - T(\alpha,\beta,\mu,\eta,\gamma,\delta)\, z^n\right\}, \qquad (6.5) \\
&= a_1 z - \sum_{n=2}^{\infty} t_n T(\alpha,\beta,\mu,\eta,\gamma,\delta)\, z^n = a_1 z - \sum_{n=2}^{\infty} a_n z^n
\end{aligned}
$$

which proves the assertion of this theorem. □

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